

Using semidirect product of (semi)groups in public key cryptography

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Abstract. In this survey, we describe a general key exchange protocol based on semidirect product of (semi)groups (more specifically, on extensions of (semi)groups by automorphisms), and then focus on practical instances of this general idea. This protocol can be based on any group or semigroup, in particular on any non-commutative group. One of its special cases is the standard Diffie-Hellman protocol, which is based on a cyclic group. However, when this protocol is used with a non-commutative (semi)group, it acquires several useful features that make it compare favorably to the Diffie-Hellman protocol. The focus then shifts to selecting an optimal platform (semi)group, in terms of security and efficiency. We show, in particular, that one can get a variety of new security assumptions by varying an automorphism used for a (semi)group extension.

1 Introduction

The area of public key cryptography started with the seminal paper [2] introducing what is now known as the Diffie-Hellman key exchange protocol.

The simplest, and original, implementation of the protocol uses the multiplicative group of integers modulo p , where p is prime and g is primitive mod p . A more general description of the protocol uses an arbitrary finite cyclic group.

1. Alice and Bob agree on a finite cyclic group G and a generating element g in G . We will write the group G multiplicatively.
2. Alice picks a random natural number a and sends g^a to Bob.
3. Bob picks a random natural number b and sends g^b to Alice.
4. Alice computes $K_A = (g^b)^a = g^{ba}$.
5. Bob computes $K_B = (g^a)^b = g^{ab}$.

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Since $ab = ba$, both Alice and Bob are now in possession of the same group element $K = K_A = K_B$ which can serve as the shared secret key.

The protocol is considered secure against eavesdroppers if G and g are chosen properly. The eavesdropper must solve the *Diffie-Hellman problem* (recover g^{ab} from g , g^a and g^b) to obtain the shared secret key. This is currently considered difficult for a “good” choice of parameters (see e.g. [8] for details).

There is an ongoing search for other platforms where the Diffie-Hellman or similar key exchange could be carried out more efficiently or where security would be based on different assumptions. This search already gave rise to several interesting directions, including a whole area of elliptic curve cryptography [17]. We also refer the reader to [10] or [11] for a survey of proposed cryptographic primitives based on non-abelian (= non-commutative) groups. A survey of these efforts is outside of the scope of the present paper; our goal here is to describe a new key exchange protocol from [4] based on extension of a (semi)group by automorphisms (or more generally, by self-homomorphisms) and discuss possible platforms that would make this protocol secure and efficient. This protocol can be based on any group, in particular on any non-commutative group. It has some resemblance to the classical Diffie-Hellman protocol, but there are several distinctive features that, we believe, give the new protocol important advantages. In particular, even though the parties do compute a large power of a public element (as in the classical Diffie-Hellman protocol), they do not transmit the whole result, but rather just part of it.

We then describe in this survey some particular instantiations of this general protocol. We start with a non-commutative semigroup of matrices as the platform, consider an extension of this semigroup by a conjugating automorphism and show that security of the relevant instantiation is based on a quite different security assumption compared to that of the standard Diffie-Hellman protocol. However, due to the nature of this security assumption, the protocol turns out to be vulnerable to a “linear algebra attack”, similar to an attack on Stickel’s protocol [16] offered in [15], albeit more sophisticated, see [9], [14]. A composition of conjugating automorphism with a field automorphism was employed in [7], but this automorphism still turned out to be not complex enough to make the protocol withstand a linear algebra attack, see [3], [14].

We therefore offer here another platform group that we believe should make the protocol invulnerable to the attacks of [3], [9], [14]. The group is a *free nilpotent p -group*, for a sufficiently large prime p . We give a formal definition of this group in Section 8; here we just say that this is a finite group all of whose elements have order dividing p^n for some fixed $n \geq 1$. As any finite group, this group is linear, but Janusz [5] showed that a faithful representation of a finite p -group, with at least one element of order p^n , as a group of matrices over a finite field of characteristic p is of dimension at least $1 + p^{n-1}$, which is too large to launch a linear algebra attack provided p itself is large enough. At the same time, to keep computation in the platform group efficient, the nilpotency class of the group has to be fairly small. We note that, in contrast, the dimension of the classical representations of finitely generated *torsion-free* nilpotent groups

in a matrix group $UT(\mathbb{Z})$ can be rather small (cf. [12]), but for torsion groups with elements of large order the situation is really different. Still, there is the usual trade-off between security and efficiency, so the following parameters have to be chosen carefully to provide for both security and efficiency: (1) the size of p ; (2) the nilpotency class of the platform group; (3) the rank (i.e., the number of generators) of the platform group. We discuss this in our Section 8.

We mention here another, rather different, proposal [13] of a cryptosystem based on the semidirect product of two groups and yet another, more complex, proposal of a key agreement based on the semidirect product of two monoids [1]. Both these proposals are very different from that of [4]. In particular, the crucial idea of transmitting just part of the result of an exponentiation appears only in [4].

Finally, we note that the basic construction (semidirect product) described in this survey can be adopted, with some simple modifications, in other algebraic systems, e.g. associative rings or Lie rings, and key exchange protocols similar to ours can be built on those.

2 Semidirect products and extensions by automorphisms

We include this section to make the exposition more comprehensive. The reader who is uncomfortable with group-theoretic constructions can skip to subsection 2.1.

We now recall the definition of a semidirect product:

Definition 1. *Let G, H be two groups, let $\text{Aut}(G)$ be the group of automorphisms of G , and let $\rho : H \rightarrow \text{Aut}(G)$ be a homomorphism. Then the semidirect product of G and H is the set*

$$\Gamma = G \rtimes_{\rho} H = \{(g, h) : g \in G, h \in H\}$$

with the group operation given by

$$(g, h)(g', h') = (g^{\rho(h')} \cdot g', h \cdot h').$$

Here $g^{\rho(h')}$ denotes the image of g under the automorphism $\rho(h')$, and when we write a product $h \cdot h'$ of two morphisms, this means that h is applied first.

In this paper, we focus on a special case of this construction, where the group H is just a subgroup of the group $\text{Aut}(G)$. If $H = \text{Aut}(G)$, then the corresponding semidirect product is called the *holomorph* of the group G . We give some more details about the holomorph in our Section 2.1, and in Section 3 we describe a key exchange protocol that uses (as the platform) an extension of a group G by a *cyclic* group of automorphisms.

2.1 Extensions by automorphisms

A particularly simple special case of the semidirect product construction is where the group H is just a subgroup of the group $\text{Aut}(G)$. If $H = \text{Aut}(G)$, then the

corresponding semidirect product is called the *holomorph* of the group G . Thus, the holomorph of G , usually denoted by $Hol(G)$, is the set of all pairs (g, ϕ) , where $g \in G$, $\phi \in Aut(G)$, with the group operation given by $(g, \phi) \cdot (g', \phi') = (\phi'(g) \cdot g', \phi \cdot \phi')$.

It is often more practical to use a subgroup of $Aut(G)$ in this construction, and this is exactly what we do in Section 3, where we describe a key exchange protocol that uses (as the platform) an extension of a group G by a cyclic group of automorphisms.

Remark 1. One can also use this construction if G is not necessarily a group, but just a semigroup, and/or consider endomorphisms (i.e., self-homomorphisms) of G , not necessarily automorphisms. Then the result will be a semigroup; this is what we use in our Section 6.

3 Key exchange protocol

In the simplest implementation of the construction described in our Section 2.1, one can use just a cyclic subgroup (or a cyclic subsemigroup) of the group $Aut(G)$ (respectively, of the semigroup $End(G)$ of endomorphisms) instead of the whole group of automorphisms of G .

Thus, let G be a (semi)group. An element $g \in G$ is chosen and made public as well as an arbitrary automorphism $\phi \in Aut(G)$ (or an arbitrary endomorphism $\phi \in End(G)$). Bob chooses a private $n \in \mathbb{N}$, while Alice chooses a private $m \in \mathbb{N}$. Both Alice and Bob are going to work with elements of the form (g, ϕ^r) , where $g \in G$, $r \in \mathbb{N}$. Note that two elements of this form are multiplied as follows: $(g, \phi^r) \cdot (h, \phi^s) = (\phi^s(g) \cdot h, \phi^{r+s})$.

1. Alice computes $(g, \phi)^m = (\phi^{m-1}(g) \cdots \phi^2(g) \cdot \phi(g) \cdot g, \phi^m)$ and sends **only the first component** of this pair to Bob. Thus, she sends to Bob **only** the element $a = \phi^{m-1}(g) \cdots \phi^2(g) \cdot \phi(g) \cdot g$ of the (semi)group G .
2. Bob computes $(g, \phi)^n = (\phi^{n-1}(g) \cdots \phi^2(g) \cdot \phi(g) \cdot g, \phi^n)$ and sends **only the first component** of this pair to Alice. Thus, he sends to Alice **only** the element $b = \phi^{n-1}(g) \cdots \phi^2(g) \cdot \phi(g) \cdot g$ of the (semi)group G .
3. Alice computes $(b, x) \cdot (a, \phi^m) = (\phi^m(b) \cdot a, x \cdot \phi^m)$. Her key is now $K_A = \phi^m(b) \cdot a$. Note that she does not actually “compute” $x \cdot \phi^m$ because she does not know the automorphism $x = \phi^n$; recall that it was not transmitted to her. But she does not need it to compute K_A .
4. Bob computes $(a, y) \cdot (b, \phi^n) = (\phi^n(a) \cdot b, y \cdot \phi^n)$. His key is now $K_B = \phi^n(a) \cdot b$. Again, Bob does not actually “compute” $y \cdot \phi^n$ because he does not know the automorphism $y = \phi^m$.
5. Since $(b, x) \cdot (a, \phi^m) = (a, y) \cdot (b, \phi^n) = (g, \phi)^{m+n}$, we should have $K_A = K_B = K$, the shared secret key.

Remark 2. Note that, in contrast with the “standard” Diffie-Hellman key exchange, correctness here is based on the equality $h^m \cdot h^n = h^n \cdot h^m = h^{m+n}$ rather than on the equality $(h^m)^n = (h^n)^m = h^{mn}$. In the “standard” Diffie-Hellman set up, our trick would not work because, if the shared key K was just the product of two openly transmitted elements, then anybody, including the eavesdropper, could compute K .

4 Computational cost

From the look of transmitted elements in the protocol in Section 3, it may seem that the parties have to compute a product of m (respectively, n) elements of the (semi)group G . However, since the parties actually compute powers of an element of G , they can use the “square-and-multiply” method, as in the standard Diffie-Hellman protocol. Then there is a cost of applying an automorphism ϕ to an element of G , and also of computing powers of ϕ . These costs depend, of course, on a specific platform (semi)group that is used with our protocol and on a specific automorphism that is used for a (semi)group extension. In our first, “toy” example (Section 5 below), both applying an automorphism ϕ and computing its powers amount to exponentiation of elements of G , which can be done again by the “square-and-multiply” method. In our example in Section 6, ϕ is a conjugation, so applying ϕ amounts to just two multiplications of elements in G , while computing powers of ϕ amounts to exponentiation of two elements of G (namely, of the conjugating element and of its inverse).

Thus, in either instantiation of our protocol considered in this paper, the cost of computing $(g, \phi)^n$ is $O(\log n)$, just as in the standard Diffie-Hellman protocol. Computational cost analysis for the platform group suggested in Section 8 is somewhat more delicate; we refer to Section 8.1 for more details.

5 “Toy example”: multiplicative \mathbb{Z}_p^*

As one of the simplest instantiations of our protocol, we use here the multiplicative group \mathbb{Z}_p^* as the platform group G to illustrate what is going on. In selecting a prime p , as well as private exponents m, n , one can follow the same guidelines as in the “standard” Diffie-Hellman.

Selecting the (public) endomorphism ϕ of the group \mathbb{Z}_p^* amounts to selecting yet another integer k , so that for every $h \in \mathbb{Z}_p^*$, one has $\phi(h) = h^k$. If k is relatively prime to $p - 1$, then ϕ is actually an automorphism. Below we assume that $k > 1$.

Then, for an element $g \in \mathbb{Z}_p^*$, we have:

$$(g, \phi)^m = (\phi^{m-1}(g) \cdots \phi(g) \cdot \phi^2(g) \cdot g, \phi^m).$$

We focus on the first component of the element on the right; easy computation shows that it is equal to $g^{k^{m-1} + \dots + k + 1} = g^{\frac{k^m - 1}{k - 1}}$. Thus, if the adversary chooses

a “direct” attack, by trying to recover the private exponent m , he will have to solve the discrete log problem twice: first to recover $\frac{k^m-1}{k-1}$ from $g^{\frac{k^m-1}{k-1}}$, and then to recover m from k^m . (Note that k is public since ϕ is public.)

On the other hand, the analog of what is called “the Diffie-Hellman problem” would be to recover the shared key $K = g^{\frac{k^{m+n}-1}{k-1}}$ from the triple $(g, g^{\frac{k^m-1}{k-1}}, g^{\frac{k^n-1}{k-1}})$. Since g and k are public, this is equivalent to recovering $g^{k^{m+n}}$ from the triple (g, g^{k^m}, g^{k^n}) , i.e., this is exactly the standard Diffie-Hellman problem.

Thus, the bottom line of this example is that the instantiation of our protocol where the group G is \mathbb{Z}_p^* , is not really different from the standard Diffie-Hellman protocol. In the next section, we describe a more interesting instantiation, where the (semi)group G is non-commutative.

6 Matrices over group rings and extensions by inner automorphisms

Our exposition here follows [4]. To begin with, we note that the general protocol in Section 3 can be used with *any* non-commutative group G if ϕ is selected to be a non-trivial inner automorphism, i.e., conjugation by an element which is not in the center of G . Furthermore, it can be used with any non-commutative *semigroup* G as well, as long as G has some invertible elements; these can be used to produce inner automorphisms. A typical example of such a semigroup would be a semigroup of matrices over some ring.

In the paper [6], the authors have employed matrices over group rings of a (small) symmetric group as platforms for the (standard) Diffie-Hellman-like key exchange. In this section, we use these matrix semigroups again and consider an extension of such a semigroup by an inner automorphism to get a platform semigroup for the general protocol in Section 3.

Recall that a (semi)group ring $R[S]$ of a (semi)group S over a commutative ring R is the set of all formal sums $\sum_{g_i \in S} r_i g_i$, where $r_i \in R$, and all but a finite number of r_i are zero.

The sum of two elements in $R[G]$ is defined by

$$\left(\sum_{g_i \in S} a_i g_i \right) + \left(\sum_{g_i \in S} b_i g_i \right) = \sum_{g_i \in S} (a_i + b_i) g_i.$$

The multiplication of two elements in $R[G]$ is defined by using distributivity.

As we have already pointed out, if a (semi)group G is non-commutative and has non-central invertible elements, then it always has a non-identical inner automorphism, i.e., conjugation by an element $g \in G$ such that $g^{-1}hg \neq h$ for at least some $h \in G$.

Now let G be the semigroup of 3×3 matrices over the group ring $\mathbb{Z}_7[A_5]$, where A_5 is the alternating group on 5 elements. Here we use an extension of the semigroup G by an inner automorphism φ_H , which is conjugation by a matrix

$H \in GL_3(\mathbb{Z}_7[A_5])$. Thus, for any matrix $M \in G$ and for any integer $k \geq 1$, we have

$$\varphi_H(M) = H^{-1}MH; \quad \varphi_H^k(M) = H^{-k}MH^k.$$

Now the general protocol from Section 3 is specialized in this case as follows.

1. Alice and Bob agree on public matrices $M \in G$ and $H \in GL_3(\mathbb{Z}_7[A_5])$. Alice selects a private positive integer m , and Bob selects a private positive integer n .
2. Alice computes $(M, \varphi_H)^m = (H^{-m+1}MH^{m-1} \dots H^{-2}MH^2 \cdot H^{-1}MH \cdot M, \varphi_H^m)$ and sends **only the first component** of this pair to Bob. Thus, she sends to Bob **only** the matrix

$$A = H^{-m+1}MH^{m-1} \dots H^{-2}MH^2 \cdot H^{-1}MH \cdot M = H^{-m}(HM)^m.$$

3. Bob computes $(M, \varphi_H)^n = (H^{-n+1}MH^{n-1} \dots H^{-2}MH^2 \cdot H^{-1}MH \cdot M, \varphi_H^n)$ and sends **only the first component** of this pair to Alice. Thus, he sends to Alice **only** the matrix

$$B = H^{-n+1}MH^{n-1} \dots H^{-2}MH^2 \cdot H^{-1}MH \cdot M = H^{-n}(HM)^n.$$

4. Alice computes $(B, x) \cdot (A, \varphi_H^m) = (\varphi_H^m(B) \cdot A, x \cdot \varphi_H^m)$. Her key is now $K_{Alice} = \varphi_H^m(B) \cdot A = H^{-(m+n)}(HM)^{m+n}$. Note that she does not actually “compute” $x \cdot \varphi_H^m$ because she does not know the automorphism $x = \varphi_H^n$; recall that it was not transmitted to her. But she does not need it to compute K_{Alice} .
5. Bob computes $(A, y) \cdot (B, \varphi_H^n) = (\varphi_H^n(A) \cdot B, y \cdot \varphi_H^n)$. His key is now $K_{Bob} = \varphi_H^n(A) \cdot B$. Again, Bob does not actually “compute” $y \cdot \varphi_H^n$ because he does not know the automorphism $y = \varphi_H^m$.
6. Since $(B, x) \cdot (A, \varphi_H^m) = (A, y) \cdot (B, \varphi_H^n) = (M, \varphi_H)^{m+n}$, we should have $K_{Alice} = K_{Bob} = K$, the shared secret key.

7 Security assumptions

In this section, we address the question of security of the protocol described in Section 6.

Recall that the shared secret key in the protocol of Section 6 is

$$K = \varphi_H^m(B) \cdot A = \varphi_H^n(A) \cdot B = H^{-(m+n)}(HM)^{m+n}.$$

Therefore, our security assumption here is that it is computationally hard to retrieve the key $K = H^{-(m+n)}(HM)^{m+n}$ from the quadruple $(H, M, H^{-m}(HM)^m, H^{-n}(HM)^n)$.

In particular, we have to take care that the matrices H and HM do not commute because otherwise, K is just a product of $H^{-m}(HM)^m$ and $H^{-n}(HM)^n$.

A weaker security assumption arises if an eavesdropper tries to recover a private exponent from a transmission, i.e., to recover, say, m from $H^{-m}(HM)^m$. A special case of this problem, where $H = I$, is the “discrete log” problem for matrices over $\mathbb{Z}_7[A_5]$, namely: recover m from M and M^m .

As we have mentioned in the Introduction, the protocol in this section was attacked in [9] and [14] by a “linear algebra attack”. This was possible partly because of the special “compact” form of the above security assumptions, and partly because the dimension of a linear representation of the platform semi-group happens to be small enough in this case for a linear algebra attack to be computationally feasible. In the following Section 8, we offer another platform that does not have these vulnerabilities.

8 Nilpotent groups and p -groups

First we recall that a *free group* F_r on x_1, \dots, x_r is the set of *reduced words* in the alphabet $\{x_1, \dots, x_r, x_1^{-1}, \dots, x_r^{-1}\}$. A reduced word is a word without subwords $x_i x_i^{-1}$ or $x_i^{-1} x_i$. The multiplication on this set is concatenation of two words, followed by canceling out all subwords $x_i x_i^{-1}$ and $x_i^{-1} x_i$ until the word becomes reduced.

It is a fact that every group that can be generated by r elements is the factor group of F_r by an appropriate normal subgroup. We are now going to define two special normal subgroups of F_r .

The normal subgroup F_r^p is generated (as a group) by all elements of the form g^p , $g \in F_r$. In the factor group F_r/F_r^p every nontrivial element therefore has order p (if p is a prime). More generally, if $n \geq 2$ is an arbitrary integer, then the order of any element of F_r/F_r^n divides n .

The other normal subgroup that we need is somewhat less straightforward to define. Let $[a, b]$ denote $a^{-1}b^{-1}ab$. Then, inductively, let $[y_1, \dots, y_{c+1}]$ denote $[[y_1, \dots, y_c], y_{c+1}]$. For a group G , denote by $\gamma_c(G)$ the (normal) subgroup of G generated (as a group) by all elements of the form $[y_1, \dots, y_c]$. If $\gamma_{c+1}(G) = \{1\}$, we say that the group G is nilpotent of nilpotency class c .

The factor group $F_r/\gamma_{c+1}(F_r)$ is called *the free nilpotent group* of nilpotency class c . This group is infinite; however, the group we define in the following subsection is finite, and we are going to recommend it as the platform for the cryptographic scheme based on a semidirect product.

8.1 Free nilpotent p -group

The group $G = F_r/F_r^{p^2} \cdot \gamma_{c+1}(F_r)$ is what we suggest to use as the platform for the key exchange protocol in Section 3.

This group, being a nilpotent p -group, is finite. Its order depends on p , c , and r . For efficiency reasons, it seems better to keep c and r fairly small (in particular, we suggest $c = 2$ or 3), while p should be large enough to make the dimension of linear representations of G so large that a linear algebra attack would be infeasible. As we have mentioned in the Introduction, a faithful representation

of a finite p -group, with at least one element of order p^n , as a group of matrices over a finite field of characteristic p is of dimension at least $1 + p^{n-1}$ [5], so in our case it is of dimension at least $1 + p$. Thus, if p is, say, a 100-bit number, a linear algebra attack is already infeasible.

At the same time, we want computation in the group G to be efficient. Also, we want transmitted elements to be in some kind of standard form, usually called a *normal form*. Here is how a normal form looks like if nilpotency class $c = 2$:

$$x_1^{\alpha_1} \cdots x_i^{\alpha_i} \cdots x_r^{\alpha_r} [x_1, x_2]^{\beta_{1,2}} \cdots [x_i, x_j]^{\beta_{i,j}} \cdots [x_{r-1}, x_r]^{\beta_{r-1,r}},$$

where α_i and $\beta_{i,j}$ are integers and in every $[x_i, x_j]$ above one has $i < j$. Different collections of α_i and $\beta_{i,j}$ produce different elements of G as long as $0 \leq \alpha_i, \beta_{i,j} < p^2$, so G in this case has at least $p^{2r+r(r-1)} = p^{r^2+r}$ elements, which is a large number even if r is fairly small. At the same time, group operations (i.e., multiplication and inversion) in G are quite efficient. Indeed, multiplying two elements in the above form essentially amounts to re-writing a product $x_1^{\alpha_1} \cdots x_r^{\alpha_r} \cdot x_1^{\alpha'_1} \cdots x_r^{\alpha'_r}$ in the normal form. This is because commutators $[x_i, x_j]$ commute with any element of G (since $c = 2$), so collecting all $[x_i, x_j]$ in the right place takes (almost) linear time in the length of an input. Now re-writing a product of powers of x_i in the normal form is not too hard either because $[x_i^a, x_j^b] = [x_i, x_j]^{ab}$ in the group G (again, since $c = 2$). Thus, re-writing will take at most quadratic time in the length of an input.

Applying an endomorphism (i.e., a self-homomorphism) ϕ given as a map $\phi(x_i) = y_i$ on the generators is efficient, too. This is due to the fact that in any group G of nilpotency class 2, one has: (1) $ab = ba$ if either a or b (or both) belong to $\gamma_2(G)$; (2) $[ab, c] = [a, c][b, c]$ and $[a, bc] = [a, b][a, c]$ for any $a, b, c \in G$; (3) $(ab)^n = a^n b^n [b, a]^{\frac{n(n-1)}{2}}$ for any $a, b \in G$. Using these identities, one can reduce $\phi(g)$ to the normal form in at most quadratic time in the length of $g \in G$, provided g itself was in the normal form.

The group G has another property useful for our purposes. We note that the subgroup $F_r^{p^r} \cdot \gamma_{c+1}(F_r)$ of F_r is, in fact, *fully invariant*, i.e., is invariant under any endomorphism of F_r . This implies that the group G has a lot of endomorphisms because any map on the generators of G can be extended (by the homomorphic property) to an endomorphism of G . Thus, if G has r generators and m elements altogether, then it has m^r endomorphisms. Even if r is very small (say, $r = 3$), this number is huge because, as we have just seen, G has at least p^{r^2+r} elements, so with a 100-bit p , we are going to have at least 2^{3600} endomorphisms. Of course, we want our endomorphism ϕ not to have short cycles (i.e., if $\phi^m = \phi^n$, then $|m - n|$ has to be quite large). This is easier to guarantee if ϕ is actually an automorphism because then we can sample from automorphisms having a large order, and these correspond to matrices from $GL_r(\mathbb{Z}_{p^2})$ that have large order. Sampling matrices of large order from that group is not completely trivial, but we leave this outside of the scope of this survey. Here we just mention that for most automorphisms of G , relevant security assumptions will not have a compact form like that in Section 7 because a product of the form $\phi^{m-1}(g) \cdots \phi^2(g) \cdot \phi(g) \cdot g$ (see the general protocol in our Section 3) typically does not simplify much.

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